4 iterative methods

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Setup: To solve $Ax=b$. Suppose $I-B$ is invertible, and that $x = Bx + c$.
Under certain conditions, starting from any vector u_0, the sequence (u_k)
$u_{k+1} = Bu_k + c$, $k \in \mathbb{N}$ converges to x.

Thm 9.1 For any square matrix B, the following conditions are equivalent.

(1) $\lim_{k\to\infty} B^k = 0$.

(2) $\lim_{k\to\infty} B^k v = 0$ $\forall$ vectors v.

(3) $\rho(B) < 1$

(4) $\|B\| < 1$, for some subordinate matrix norm $\| \ \|$.

proof. (1) → (2) Assume $\lim_{k\to\infty} B^k = 0$, then $\lim_{k\to\infty} \|B^k v\| \le \lim_{k\to\infty} \|B^k\| \, \|v\| = 0$

$$\Rightarrow \lim_{k\to\infty} B^k v = 0.$$

(2) → (3) Assume $\lim_{k\to\infty} B^k v = 0$. If $\rho(B) \ge 1$, then $\exists\, u \ne 0$ and λ
s.t. $Bu = \lambda u$, $|\lambda| = \rho(B) \ge 1$.

But then $(B^k u) \not\to 0$ because $B^k u = \lambda^k u$ and $|\lambda|^k \ge 1$.

Thus, $\rho(B) < 1$.

(3) → (4) Assume $\rho(B) < 1$. By Prop. 8.9/8.12, $\forall\, \varepsilon > 0$, $\exists$ a subordinate matrix norm $\| \ \|$ s.t. $\|B\| \le \rho(B) + \varepsilon$.
Choose ε small enough that $\rho(B) + \varepsilon < 1$.

(4) → (1) Assume $\|B\| < 1$ for some subordinate matrix norm $\| \ \|$.
Then $\|B^k\| \le \|B\|^k \Rightarrow \lim_{k\to\infty} \|B^k\| = 0 \Rightarrow \lim_{k\to\infty} B^k = 0$.

Prop 9.1/9.2 For every square matrix $B \in \mathbb{C}^{n\times n}$ and every matrix norm $\| \ \|$,

$$\lim_{k\to\infty} \|B^k\|^{1/k} = \rho(B).$$

proof. Recall that $\rho(B) \le \|B\|$ and $\rho(B) = (\rho(B^k))^{1/k}$ (think eigenvalues)

$\Rightarrow \rho(B) \le \|B^k\|^{1/k}$ $\forall\, k \ge 1$

proof. Recall that $\rho(B) \le \|B\|$ and $\rho(B) = (\rho(B^k))^{1/k}$

$$\Rightarrow \rho(B) \le \|B^k\|^{1/k} \quad \forall k \ge 1,$$

$$\Rightarrow \rho(B) \le \lim_{k\to\infty} \|B^k\|^{1/k}.$$

Need to show $\forall \varepsilon > 0$, $\exists N(\varepsilon)$ s.t.

$$\|B^k\|^{1/k} \le \rho(B) + \varepsilon \quad \text{for all } k \ge N(\varepsilon).$$

$\forall \varepsilon > 0$, let $B_\varepsilon = \dfrac{B}{\rho(B)+\varepsilon}$. Then $\|B_\varepsilon\| < 1$.

$$\Rightarrow \lim_{k\to\infty} B_\varepsilon^k = 0.$$

$$\Rightarrow \exists N(\varepsilon) \text{ s.t. } \forall k \ge N(\varepsilon),$$

$$\|B_\varepsilon^k\| = \frac{\|B^k\|}{(\rho(B)+\varepsilon)^k} \le 1$$

$$\Rightarrow \|B^k\|^{1/k} \le \rho(B) + \varepsilon. \qquad \square$$

Thm 9.2/9.3 Given a system $u = Bu + c$, with a unique solution $\tilde{u}$ s.t. $\tilde{u} = \tilde{x}$, the unique solution to $Ax = b$, if $I - B$ is invertible, then the following are equivalent,

(1) Let $u_{k+1} = Bu_k + c$. $\lim_{k\to\infty} u_k = \tilde{u}$. (i.e. the iterative method is convergent)

(2) $\rho(B) < 1$.

(3) $\|B\| < 1$, for some subordinate matrix norm $\|\ \|$.

proof. Define the error vector $e_k = u_k - \tilde{u}$.

Clearly convergence is equivalent to $\lim_{k\to\infty} e_k = 0$.

Claim: $e_k = B^k e_0$, $k \ge 0$, where $e_0 = u_0 - \tilde{u}$.

proof. By induction. Base case $k=0$ is trivial.

Then $u_{k+1} = Bu_k + c$

$$\Rightarrow u_{k+1} - \tilde{u} = Bu_k + c - \tilde{u}$$

$$= Bu_k - B\tilde{u}$$

$$\Rightarrow \quad u_{k+1} - \tilde{u} = Bu_k + c - u$$
$$= Bu_k - B\tilde{u}$$
$$= B(u_k - \tilde{u})$$
$$= Be_k = BB^k e_0 = B^{k+1} e_0,$$

proving the induction step. $\square$

Thus, the iterative method converges iff $\lim_{k\to\infty} B^k e_0 = 0$.

Now just apply Thm 9.! $\square$

Prop. 9.2/9.4 Let $\| \; \|$ be any norm, let $B \in \mathbb{C}^{n\times n}$ be a matrix such that $I - B$ is invertible, and let $\tilde{u}$ be the unique soln of $u = Bu + c$.

(1) If (u_k) is any sequence defined iteratively by

$$u_{k+1} = Bu_k + c \quad , k \in \mathbb{N},$$

then

$$\lim_{k\to\infty} \left[\sup_{\|u_0 - \tilde{u}\| = 1} \|u_k - \tilde{u}\|^{1/k} \right] = \rho(B)$$

(2) Let B_1 and B_2 be two matrices such that $I - B_1$ and $I - B_2$ are invertible. Assume that both $u = B_1 u + c_1$ and $u = B_2 u + c_2$ have the same unique soln $\tilde{u}$, and consider two sequences (u_k) and (v_k) defined inductively by

$$u_{k+1} = B_1 u_k + c_1$$
$$v_{k+1} = B_2 v_k + c_2$$

with $u_0 = v_0$. If $\rho(B_1) < \rho(B_2)$, then $\forall \varepsilon > 0$, there is some integer $N(\varepsilon)$, s.t. for all $k \geq N(\varepsilon)$, we have

$$\sup_{\|u_0 - \tilde{u}\| = 1} \left[\frac{\|v_k - \tilde{v}\|}{\|u_k - \tilde{u}\|} \right]^{1/k} \geq \frac{\rho(B_2)}{\rho(B_1) + \varepsilon}$$

proof. Let $\| \; \|$ be the subordinate matrix norm. Recall that

proof. Let $\| \ \|$ be the subordinate matrix norm. Recall that

$$u_k - \tilde{u} = B^k e_0 ,$$

with $e_0 = u_0 - \tilde{u}$. For every $k \in \mathbb{N}$,

$$\|B^k\| = \sup_{\|e_0\|=1} \|B^k e_0\|,$$

$$\Rightarrow \quad \sup_{\|e_0\|=1} \|B^k e_0\|^{\frac{1}{k}} = \|B^k\|^{1/k}$$

By Prop. 9.1/9.2, $\lim_{k\to\infty} \|B^k\|^{\frac{1}{k}} = \rho(B)$

$$\Rightarrow \quad \lim_{k\to\infty} \left[\sup_{\|e_0\|=1} \|B^k e_0\|^{\frac{1}{k}} \right] = \rho(B)$$

$$\lim_{k\to\infty} \left[\sup_{\|u_0 - \tilde{u}\|=1} \|u_k - \tilde{u}\|^{\frac{1}{k}} \right] = \rho(B),$$ proving part (1)

Note that
$$u_k - \tilde{u} = B_1^k e_0$$
$$v_k - \tilde{u} = B_2^k e_0 , \quad \text{with } e_0 = u_0 - \tilde{u} = v_0 - \tilde{u}.$$

By Prop 9.1/9.2, $\forall \varepsilon > 0$, $\exists N(\varepsilon)$ s.t. if $k \geq N(\varepsilon)$, then

$$\|B_1^k\|^{\frac{1}{k}} \leq \rho(B_1) + \varepsilon$$

$$\Rightarrow \quad \sup_{\|e_0\|=1} \|B_1^k e_0\|^{\frac{1}{k}} \leq \rho(B_1) + \varepsilon$$

Furthermore, $\rho(B_2)^k = \rho(B_2^k) \leq \|B_2^k\| = \sup_{\|e_0\|=1} \|B_1^k e_0\|$

$$\Rightarrow \quad \exists \ e_0 = e_0(k) \text{ for every } k \geq N(\varepsilon)$$
$$\|e_0\| = 1 \text{ and } \|B_2^k e_0\|^{\frac{1}{k}} \geq \rho(B_2).$$

Then
$$\sup_{\|u_0 - \tilde{u}\|=1} \left[\frac{\|v_k - \tilde{u}\|}{\|u_k - \tilde{u}\|} \right]^{1/k} = \sup_{\|e_0\|=1} \frac{\|B_2^k e_0\|^{1/k}}{\|B_1^k e_0\|^{1/k}} \geq \frac{\rho(B_2)}{\rho(B_1) + \varepsilon} .$$

Thus, $\rho(B) < 1$ implies convergence, and the smaller $\rho(B)$, the faster the method converges.

faster the method converges.

In worst case, the error vector $e_k = B^k e_0$ behaves like $(\rho(B))^k$.

Consider the linear system $Ax = b$, with A invertible.

Suppose $A = M - N$, where M is invertible and "easy to invert"

(i.e. close to diagonal or triangular, perhaps by blocks),

then $Au = b \iff Mu = Nu + b$

$$\iff u = M^{-1}Nu + M^{-1}b$$

Let $B = M^{-1}N$ and $c = M^{-1}b$, and we get a system $u = Bu + c$, as described before.

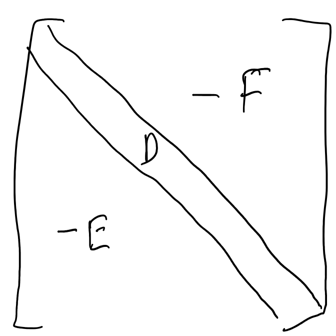
Practically, solving $Mu_{k+1} = Nu_k + b$, $k \geq 0$ is faster than inverting M.

Three iterative methods

Let $A = D - E - F$, where D is $\text{diag}(A)$,

E only has nonzero entries below the diagonal

F only has nonzero entries above the diagonal.

i.e. $A =$ 

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Jacobi method: Assume $a_{ii} \neq 0 \ \forall i$. (diagonal entries all nonzero)

Let $M = D$

$N = E + F$.

$$\Rightarrow B = M^{-1}N = D^{-1}(E+F) = I - D^{-1}A.$$

Notation: Jacobi's matrix $J = I - D^{-1}A$.

Repeatedly solve $Du_{k+1} = (E+F)u_k + b$, $k \geq 0$.

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Gauss-Seidel: Let $M = D - E$
$N = F$.

$$\Rightarrow B = M^{-1}N = (D-E)^{-1}F$$

Notation: matrix of Gauss-Seidel $\mathcal{L}_1 = (D-E)^{-1}F$.

Relaxation: Let $M = \dfrac{D}{\omega} - E$

$$N = \frac{1-\omega}{\omega} D + F,$$ where $\omega \neq 0$, $\omega \in \mathbb{R}$, is the parameter of relaxation

Relaxation matrix $\mathcal{L}_\omega = \left(\dfrac{D}{\omega} - E\right)^{-1}\left(\dfrac{1-\omega}{\omega} D + F\right) = (D - \omega E)^{-1}((1-\omega)D + \omega F)$.

Repeatedly solve $(D - \omega E)u_{k+1} = ((1-\omega)D + \omega F)u_k + \omega b$

$$\Leftrightarrow \quad Du_{k+1} = Du_k - \omega(Du_k - Eu_{k+1} - Fu_k - b)$$

Prop. 9.3/9.5: Let A be any Hermitian pos. def. matrix, written as $A = M - N$, with M invertible.
Then $M^* + N$ is Hermitian, and if it is pos. def., then $\rho(M^{-1}N) < 1$, so the iterative method converges.

proof. Since $M = A + N$ and A is Hermitian, $A^* = A$, so

$$M^* + N = A^* + N^* + N = A + N + N^* = M + N^* = (M^* + N)^*$$

$\Rightarrow M^* + N$ is Hermitian.

Consider the vector norm $\| \ \|$ defined by $\|v\| = \sqrt{v^* A v}$, and its subordinate matrix norm.

Claim: $\|M^{-1}N\| < 1$

proof. $\|M^{-1}N\| = \|I - M^{-1}A\| = \sup\limits_{\|v\| = 1} \|v - M^{-1}Av\|$

Let $w = M^{-1}Av$. $\Leftrightarrow v = A^{-1}Mw$. Then

Let $w = M^{-1}Av$. $\iff$ $v = A^{-1}Mw$. Then

$$\|v - w\|^2 = (v-w)^* A(v-w)$$
$$= \|v\|^2 - v^*Aw - w^*Av + w^*Aw$$
$$= 1 - w^*M^*w - w^*Mw + w^*Aw$$
$$= 1 - w^* \underbrace{(M^* + N)}_{\text{pos. def.}} w$$

If $w \neq 0$, then $w^*(M^* + N)w > 0$

$\Rightarrow$ if $\|v\| = 1$, then $\|v - M^{-1}Av\| < 1$.

But $v \mapsto \|v - M^{-1}Av\|$ is continuous because it is a composition of continuous functions, and therefore has a maximum on the compact subset $\{v \in \mathbb{C}^n \mid \|v\| = 1\}$

$$\Rightarrow \sup_{\|v\|=1} \|v - M^{-1}Av\| < 1. \quad \boxed{\diagup}$$

Thus, $\|M^{-1}N\| < 1$, so by Thm 9.1, the iterative method converges $\boxed{\diagup}$

$\Rightarrow \rho(M^{-1}N) < 1$

(Ostrowski-Reich Thm)

Thm 9.3/9.6 If $A = D - E - F$ is Hermitian positive def., and if $0 < \omega < 2$, then the relaxation method converges. This also holds for a block decomposition of A.

proof. Recall $A = M - N$ with $M = \frac{D}{\omega} - E$, $N = \frac{1-\omega}{\omega} D + F$.

Note $D^* = D$, $E^* = F$, and $\omega \neq 0$ is real, so

$$M^* + N = \frac{D^*}{\omega} - E^* + \frac{1-\omega}{\omega} D + F$$

$$= \frac{2-\omega}{\omega} \cdot D$$

A is Hermitian pos. def. $a_{ii} \neq 0 \Rightarrow D$ has no zeros on

$$-\frac{\;}{\omega}\cdot D$$

But because A is Hermitian pos def., $a_{ii} \neq 0 \Rightarrow D$ has no zeros on the diagonal.

$\Rightarrow$ For $\omega \in (0,2)$, $\frac{2-\omega}{\omega} D$ is Hermitian pos def.

The same argument carries through if D consists of diagonal blocks of A.

So $M^* + N$ is Hermitian pos. def. $\Rightarrow$ the relaxation method converges. $\square$

Prop. 9.4/9.7 Given any matrix $A = D - E - F$, with A and D invertible, for any $\omega \neq 0$, we have

$$\rho(\mathcal{L}_\omega) \geq |\omega - 1|,$$

where $\mathcal{L}_\omega = \left(\frac{D}{\omega} - E\right)^{-1}\left(\frac{1-\omega}{\omega} D + F\right)$. Therefore, the relaxation method doesn't converge unless $\omega \in (0,2]$. If $\omega \in \mathbb{C}$, then we must have $|\omega - 1| < 1$ for relaxation to converge.

proof. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of $\mathcal{L}_\omega$.

Then $$\lambda_1 \cdots \lambda_n = \det(\mathcal{L}_\omega) = \frac{\det\left(\frac{1-\omega}{\omega} D + F\right)}{\det\left(\frac{D}{\omega} - E\right)} = (1-\omega)^n$$

$$\Rightarrow \rho(\mathcal{L}_\omega) \geq |\lambda_1 \cdots \lambda_n|^{\frac{1}{n}} = |\omega - 1|$$ $\square$

So, the last couple theorems give information about convergence of Gauss-Jordan and the method of relaxation.

You can do similar analyses for specific Jacobi matrices, and there are relationships between Jacobi and Gauss Seidel (e.g. for tridiagonal matrices), but if you're interested, you should take a numerical analysis class